FOURIER SERIES

Trigonometric, Polar and Complex/Exponential Transforms

EEEN 462 – ANALOGUE COMMUNICATION SYSTEMS

Friday, 19 December 2025

WHERE ARE WE IN THE SYLLABUS?

Course content

Signal Analysis: Signal representation in time and frequency domain. Generalized expansion in complete orthonormal sets. Interpretation of signals as vectors in signal space. Fourier series. Fourier and Laplace Transforms. Signal spectra and their properties. Rayleigh theorem. Sampling theorem for baseband and passband signals. Power and spectral energy densities. The DFT. the FFT. The Hilbert Transform. Deterministic signals through linear systems. Analog Modulation: Need for modulation. Linear Modulation: AM, DSB, VSB, SSB.Angle modulation FM & PM. Demodulators for AM and FM signals. Frequency division multiplexing. Noise in analog modulation: Effects of noise in AM and FM systems. Demodulator performance in the presence of noise. Pre-emphasis and de-emphasis filtering threshold in FM system. Comparison of system performance in noise (AM & FM).

WHAT IS FOURIER SERIES?

- **A Fourier series** is a way to represent a periodic function as a sum of sine and cosine functions **or equivalently**, as a sum of complex exponentials, each with different frequencies and amplitudes.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} a_n sinnx$$

OR

- **A Fourier series** is an expansion of a periodic function f(x) as an infinite sum of sines and cosines **or equivalently**, as a sum of complex exponentials, each with different frequencies and amplitudes.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{jnx}$$

CONDITIONS FOR EXISTANCE OF FOURIER SERIES

A **continuous-time Fourier series** exists for a function x(t) when the following conditions (called the **Dirichlet's conditions)** are satisfied:

1. **Periodicity:** x(t) must be periodic (repeats over a fixed interval, T).
2. **Finite Discontinuities:** x(t) possesses a finite number of discontinuities in the period, T
3. **Finite Extrema:** x(t) has a finite number of maxima and minima in the period, T
4. Absolute Integrability: The integral of the absolute value of the function over one period must be finite, i.e $\int_0^T [x(t)]\,\mathrm{dt} < \infty$

TROGONOMETRIC FOURIER SERIES

A periodic signal x(t) can be expressed in the form of trigonometric Fourier series comprising the following sine and cosine terms:

$$x(t) = a_0 + a_1 cos\omega_o t + a_2 \cos(2\omega_o t) + a_n \cos(n\omega_o t) + b_1 sin\omega_o t + b_2 \sin(2\omega_o t) + b_n \sin(n\omega_o t) + ...$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(\omega_0 nt) + b_n sin(\omega_0 nt)$$

Where

$$t_o < t < t_0 + T$$

$$T = \frac{2\pi}{\omega_0}$$

ω_0 is the fundamental frequency and $2\omega_o$, $3\omega_o$ are the harmonics

TROGONOMETRIC FOURIER SERIES FOR PERIODIC FUNCTION

For a periodic function x(t) with period T, we can write the Fourier coefficients as follows:

$$a_o = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

The average value of the DC component of x(t)

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(\omega_0 nt)\,dt$$

$$= \frac{2}{T}\int_{0}^{T} x(t)\cos(\omega_0 nt)\,dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(\omega_0 nt)\,dt$$

$$= \frac{2}{T}\int_{0}^{T} x(t)\sin(\omega_0 nt)\,dt$$

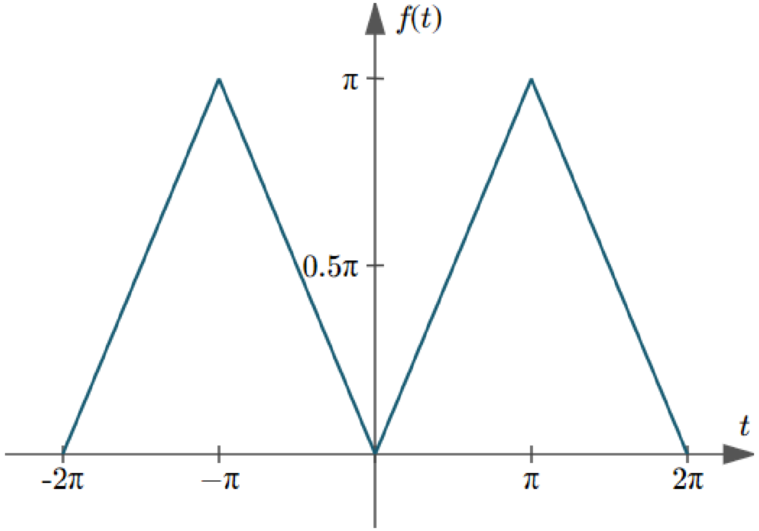
SYMMETRY CONDITIONS FOR INTEGRALS

Even Function

For even function,

x(-t) = x(t)

$$\int_{-a}^{a} x(t)dt = 2\int_{0}^{a} x(t)dt$$



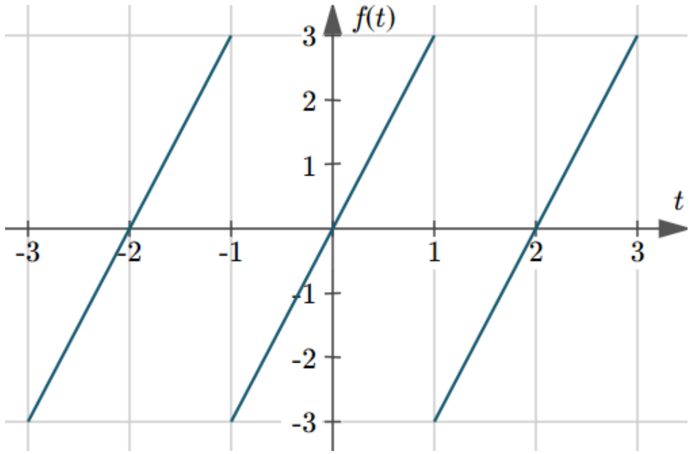
(a) Even triangular function

Odd Function

For odd function,

x(-t) = -x(t)

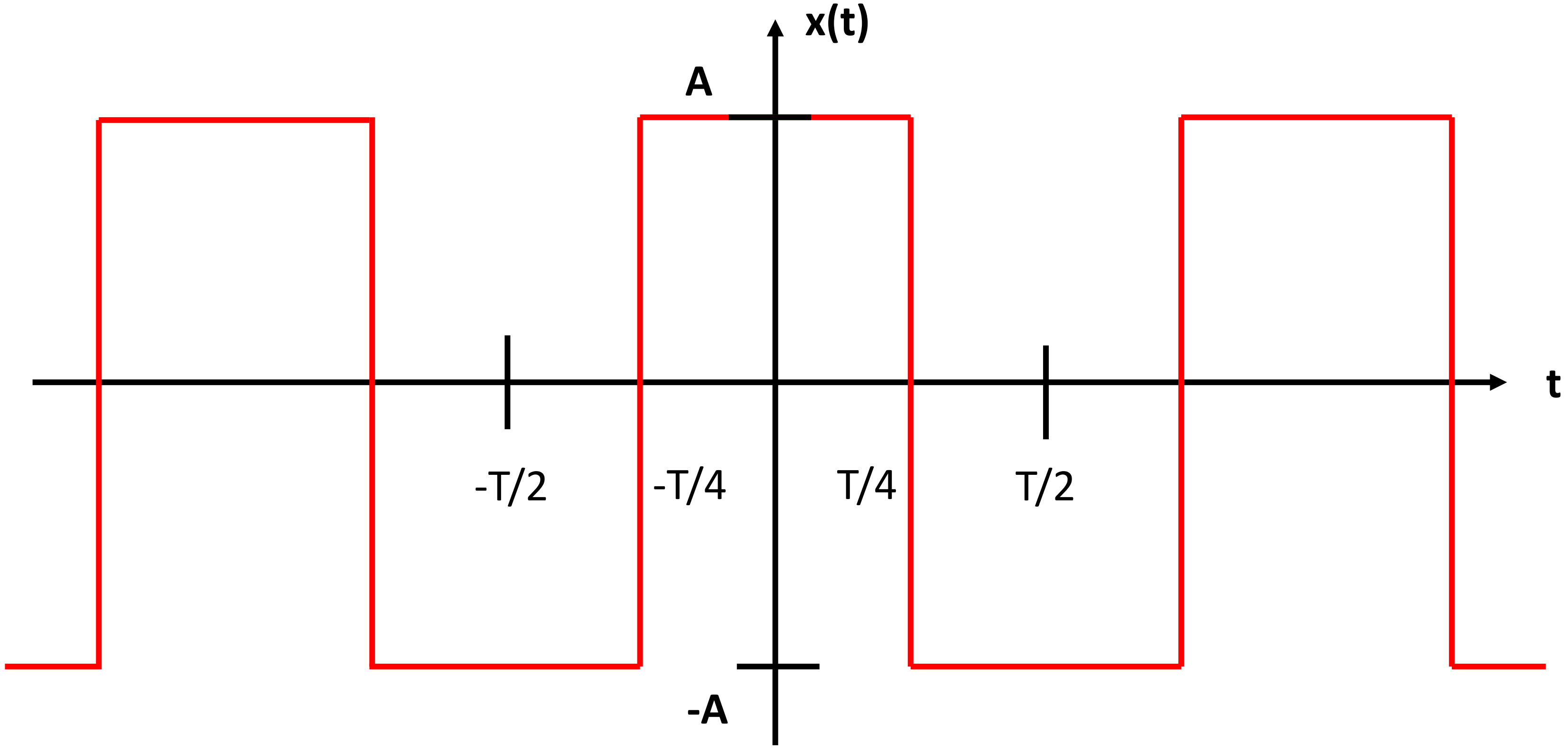
$$\int_{-a}^{a} x(t)dt = 0$$



(b) Odd sawtooth function

WORKED EXAMPLE -1

Obtain the Fourier series representation of a periodic square wave shown below.

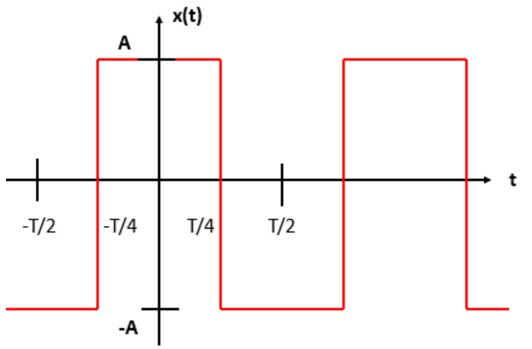


WORKED EXAMPLE -1

The expression for Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(\omega_0 nt) + b_n sin(\omega_0 nt)$$

1. Since the x(t) is symmetrical about the x-axis, i.e x(t) = -x(t), only the cosine terms are present.
2. The waveform is symmetrical about the horizontal axis, therefore the DC term a_0 = 0.



The expected Fourier series should therefore be of the form:

$$x(t) = \sum_{n=1}^{\infty} a_n cos(\omega_0 nt)$$

We can write

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(\omega_0 nt)\,dt = \frac{2}{T}\int_{-T/4}^{T/4} Acos(\omega_0 nt) + \frac{2}{T}\int_{-T/4}^{-T/2} -Acos(\omega_0 nt) + \frac{2}{T}\int_{T/4}^{T/2} -Acos(\omega_0 nt)$$

$$= \frac{8A}{n\omega_0 T} sin\left(\frac{n\omega_0 T}{4}\right) - \frac{4A}{n\omega_0 T} sin\left(\frac{n\omega_0 T}{2}\right)$$

Substituting $\omega_0 = \frac{2\pi}{T}$ the 2nd term becomes zero for all integer values of n, and we remain with

$a_n = \frac{4A}{n\pi} sin\left(\frac{n\pi}{2}\right)$ showing a_n is non-zero for odd values of n

or

$$x(t) = \frac{4A}{\pi}\left(cos\omega_0 t - \frac{1}{3}cos3\omega_0 t + \frac{1}{5}cos5\omega_0 t + \ldots\ldots\right)$$

POLAR FOURIER SERIES REPRESENTATION

Polar Fourier representation (Also called Compact Trigonometric Form) is a modified form of trigonometric Fourier series which we have seen is given by:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(\omega_0 nt) + b_n sin(\omega_0 nt)$$

or

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}}\cos(\omega_0 nt) + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} sin(\omega_0 nt)\right]$$

If

$\frac{a_n}{\sqrt{a_n^2+b_n^2}} = \cos\varphi_n \;\; and \frac{b_n}{\sqrt{a_n^2+b_n^2}} = \sin\varphi_n$ So that $\tan\varphi_n = \frac{b_n}{a_n}$

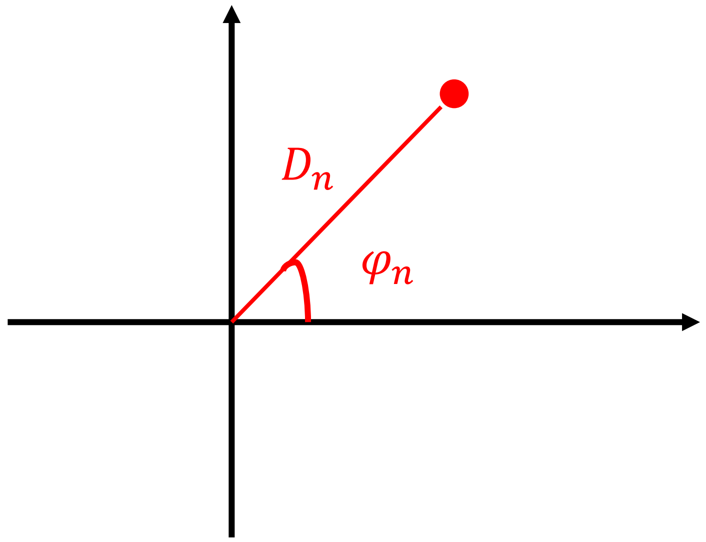
We can then write

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}(\cos(\omega_0 nt - \varphi_n)$$

where

$D_n = \sqrt{a_n^2 + b_n^2}$ is referred to as spectral amplitude of the function x(t)

φ_n is referred to as the phase spectrum of the function x(t)



EXPONENTIAL FORM FOURIER SERIES /01

The complex form of Fourier series is simpler and more compact. It is therefore more widely used in signal analysis.

From trigonometric Fourier Series,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(\omega_0 nt) + b_n sin(\omega_0 nt)$$

We know from Euler Identity that

$$e^{i\theta} = cos\theta + jsin\theta \text{ or } e^{-i\theta} = cos\theta - jsin\theta$$

We can write

$$\cos \omega_0 nt = \frac{1}{2}\left(e^{j\omega_o nt} + e^{-j\omega_0 nt}\right)$$

$$\sin \omega_0 nt = \frac{1}{2}\left(e^{j\omega_o t} - e^{-j\omega_0 t}\right)$$

Substituting, we get

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{j\omega_o nt} + e^{-j\omega_0 nt}\right) + \frac{b_n}{2}\left(e^{j\omega_o t} - e^{-j\omega_0 t}\right)$$

EXPONENTIAL FORM FOURIER SERIES /02

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{j\omega_o nt} + e^{-j\omega_0 nt}\right) + \frac{b_n}{2}\left(e^{j\omega_o t} - e^{-j\omega_0 t}\right)$$

We can write

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{j\omega_o nt} + \sum_{n=-\infty}^{-1} C_{-n} e^{-j\omega_o nt}$$

Which can be rewritten as:

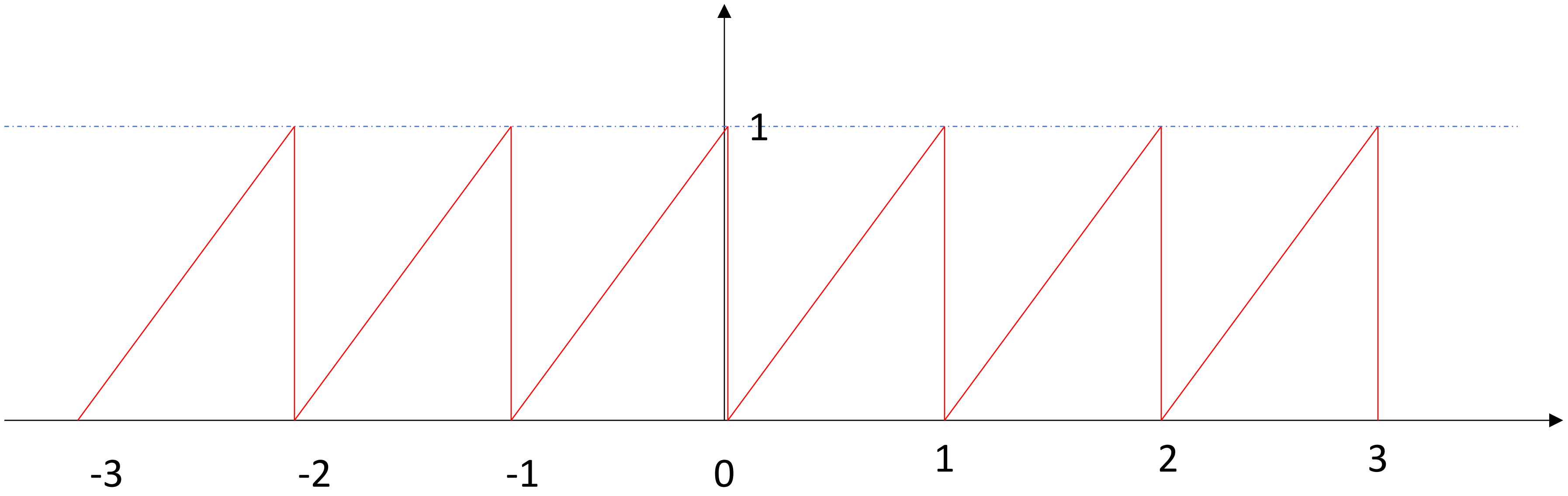
$$x(t) = c_0 + \sum_{n=-\infty}^{\infty} c_n e^{j\omega_o nt}$$

Where

$$c_n = \frac{1}{T}\int_{-\pi/2}^{\pi/2} x(t) e^{-j\omega_0 nt}$$

WORKED EXAMPLE

Find the exponential-form Fourier series of the sawtooth waveform shown below. Plot the magnitude and phase.



WORKED EXAMPLE / 01

Over one period, the signal may be represented as:

x(t) = t for $0 < t < 1$ and the period T = 1

The Fourier series coefficients of the signal x(t) are therefore

$C_n = \frac{1}{T}\int_0^T t\, e^{-i2\pi nt/T} dt = \int_0^1 te^{-i2\pi nt}\, dt$

Using the integration identity

$$\int xe^{ax}dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$$

Taking a = $-j2\pi n$ we get

$C_n = \frac{-j}{2\pi n}$ for $n \neq 0$

And

$$C_0 = \int_0^T x(t)dt = \int_0^1 tdt = \frac{1}{2}$$

WORKED EXAMPLE / 02

For $n \neq 0$

$$|C_n| = \sqrt{0 + \frac{1}{(2\pi n)^2}} = \frac{1}{2\pi n}$$

For n = 0

$$C_0 = \frac{1}{2}$$

The phase

$$\theta_n = \arg(C_n) = \pm 90^o$$